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Article

Zermelo Game: All or None?

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Abstract

This paper presents the software *Zermelo Game*, a free online game designed to support educational activities related to sets and quantifiers. The presentation of the game is accompanied by pilot tests concerning its application in various contexts: from the beginning of primary school to the end of high school. We will describe the mathematical and educational framework behind the software, how the software works and how it was used during classroom activities. The trial runs suggest some findings which merit future research endeavours. The most important findings, in our view, concern the role of the game in promoting collaborative and reflective processes, *i.e.*, the discussions among students themselves as well as between teachers and students regarding why a certain statement is or is not true.

1. Introduction

The role that computer games can have in education, and particularly in mathematics education, is increasingly studied by researchers. In 2015 the *International Journal of Serious Games* dedicated a special issue to mathematics education (Vol. 2 n. 4). In the Editorial, Kiili, Devlin, and Multisilta [1] identify some important characteristics for mathematics learning games, namely that these should be founded on theoretically sound principles, integrate mathematics directly into the gameplay, rely on good pedagogical practices, and truly utilize the possibilities that game technologies provide for learning. In the same year, the book *Digital Games and Mathematics Learning* [2] explores the influence and impact of digital games on young students' mathematics engagement, particularly focusing on learning situations beyond classrooms. The *Handbook of Research on International Approaches and Practices for Gamifying Mathematics* [3] investigates the great challenge consisting in the design of materials for mathematics content learning, and the potential of game-based learning as a dynamic way to engage and motivate learners. It also addresses the possible aid that a computer game can give in bilingual or plurilingual contexts.

In this work, we present the software *Zermelo Game*, a free online game designed to support educational activities related to logic, sets, and quantifiers. The game was designed by Luigi Bernardi and developed by Giulia Balboni, Martina Carbone, Mattia Sanchioni, and Jacopo Zuliani, employing classic online game elements to enhance student motivation and engagement in learning logic through captivating visuals, adjustable difficulty levels, self-competition and rankings, self-paced learning, and instant feedback upon errors. The game has been used in trial runs carried out in various contexts from the beginning of primary school to the end of high school.

Within primary school, *Zermelo Game* has been integrated with the *Zermelo* educational path. Through the description of various sets drawn on boards—containing numbers, figures, animals, etc.—the *Zermelo* educational path aims to develop students' sense of observation, as well as their ability to express and verify properties of elements of certain sets. Correct and incorrect descriptions are proposed and requested, with the aid of Smullyan's characters: the *knight* who always tells the truth, and the *knave* who always lies [4]. The activity leads to the use and analysis of the words *all*, *at most*, *at least*, and *none*, as well as their negation. In the path taken with high school students, the game is additionally alternated with moments of work on quantifiers, argumentation, and deduction.

A more detailed description of the activities carried out in primary school classrooms can be found at [5][1]. In this paper, we will focus on *Zermelo Game*, describing the mathematical framework behind the software, how the software works, and how it was used during classroom activities.

The overall objectives of this research project—the initial results of which are presented here—are varied and closely interconnected. Firstly, we wish to study how the class responds to the teaching and learning of logical quantifiers and their symbolism, both emotionally and mathematically, identifying emerging difficulties and key conceptual knots to untangle. A further aim is to explore whether and how gameplay facilitates (or at least does not hinder) verbalization, and if this verbalization can prove beneficial in game performance.

Since the game does not introduce elements external to those theoretically suggested by Game Semantics (see Section 2.3), the educational value of certain theoretical elements can be examined, particularly the role of the witness in promoting a conceptual change[2] regarding argumentation.

The results of the *Zermelo Game* are deeply interwoven with those of the educational path, making it challenging to determine which practices contributed to specific learning outcomes. The primary role of the game in reinforcing the learning of the entire educational path should be verified through pre- and post-tests and control classes. For the moment, partial answers to the research questions presented above are provided through the field notes of teachers and researchers, audio-video recordings, responses from an open-ended anonymous questionnaire administered to the secondary school students involved in the experiment, and data provided by the literature.

The collected data were analyzed using a qualitative methodology. The analysis involves searching for and observing moments of interaction between students, as well as between students and the teacher, along with personal opinions of students and teachers regarding the game, quantifiers, and logic in general.

2. Mathematical Foundations of *Zermelo Game*

2.1 Games of Logic in Mathematics Education

One of the first games explicitly dedicated to logic is the classic board game *Game of Logic* by Lewis Carroll [7], designed for learning to solve syllogisms, *i.e.*, statements containing quantifiers with one or more intermediate terms in common. The player has tokens of two colors at their

[1]Currently, the description of the educational path is available exclusively in Italian. It will soon be supplemented with descriptions in English and French.

[2]In the sense proposed in [6].

disposal, which stand for *exists* or *none*, that they place on the diagram to indicate the existence or absence of elements in certain sets: essentially, it is a matter of interpreting appropriate set inclusions giving them logical meaning. Later, Carroll made his game more complex and advanced with *Symbolic Logic*, building a tool for the resolution of syllogisms [8]. The fact that the author used the pseudonym Carroll, which he chose for his famous children's books, suggests that he wanted to emphasize the playful aspect of it.

Another example of a logic game is *Tarski's World* [9], a computer-based introduction to first-order logic. The computer program introduces the semantics of logic through games in which three-dimensional worlds are populated with various geometric figures that are used by the player to test the truth or falsehood of first-order logic sentences. Dubinsky and Yiparaki [10] discuss the use of quantifier games as a pedagogical tool to help students understand statements with alternated quantifiers: two players work on a sentence containing a universal and an existential quantifier, choosing values of two variables in two given sets and trying to verify a given relation among them. So, given some x chosen by player A, player B looks for some y such that a certain relation $R(x,y)$ holds. In [11], Bernardi describes the online software *Bul Game*, designed to support didactic activities about logic and its connections with natural language. The aim of the game is to make correct choices based on statements made by knights, who always tell the truth, and by knaves, who always lie. The statements involve predicates, connectives, negation, and implication. To conclude, we remark that when speaking of logical games, we often find reference to puzzles or brain teasers, not really linked to formal logic nor mathematics, although this does not mean that they cannot have an educational and logical value. For instance, Bottino and Ott [12] analyze the use of computer mind games to develop strategic and reasoning abilities in primary school students.

To our knowledge, *Zermelo Game* is the first software to explicitly integrate Game Semantics, giving the concept of a witness the central role it has in proof-theory. Additionally, given the simplicity of some levels of the game, it is the only one among the mentioned logic games that can be played in primary school or even earlier.

2.2 Quantifiers in Mathematics Education

The symbols $\forall$ (to be read as *for all*) and $\exists$ (to be read as *exists*) are known in mathematical logic as quantifiers. As the word suggests, a quantifier indicates the "quantity" of elements with a certain property. It is also possible to express the concept *none* using quantifiers: asserting that "no animal can fly" is equivalent to asserting that "all animals cannot fly". More generally, saying "no one has property A" is equivalent to saying "everyone does not have property A" or even "there does not exist an element which satisfies the property A". Gottlob Frege (1848-1925) was the first to use a quantifier in a formal mathematical context [13], introducing quantified variables. Giuseppe Peano (1858-1932) then introduced specific notations for quantifiers, in particular the symbol $\exists$ [14]. In 1935, Gerhard Gentzen (1909-1945) introduced the symbol $\forall$ [15]—modifying Peano's less clear notation for the universal quantifier—which became canonical in the 1960s. Indeed Tarsky, in 1959, still uses other symbols [16].

Most of the games mentioned in the previous section deal with quantifiers and aim to support their learning. Indeed, the difficulty of working with quantifiers, once they begin to appear in mathematics education, is emphasized by many authors. In particular, the relationship quantifiers have with natural language is not always clear, sometimes resulting in a barrier to learning the quantifiers themselves. Most authors refer, in their research, to high school or university—that is, the point at which difficulties in argumentation [17] or in working with statements involving multiple quantifiers [10, 18] become evident. Even though quantifiers, more or less explicitly, are present from the early years of mathematical education, the practical school environment does not recognize the need to explore them independently from their context of use, and mottos such as ”logic is learned automatically by teaching other mathematical disciplines” are common among teachers (see [19]). Dubinsky and Yiparaki [10] studied students’ interpretations of statements involving both universal and existential quantifiers linking these to everyday discourse. They found that students do not have a strong understanding of quantifiers in natural language either, particularly concerning statements in which the existential quantifier precedes the universal quantifier. The two authors believe that students interpret statements containing quantifiers subjectively; in other words, students interpret such statements from within a personal context that they believe is implicit and shared with the interlocutor. It is therefore important, in their opinion, to avoid situations familiar to students and focus on the syntactic aspect of the statement. Bardelle [20] carried out a study with about 300 Italian science undergraduates, concerning the negation of quantifiers, showing that everyday communication heavily affects the interpretation of a variety of statements. Indeed, in current language, the meaning of some syntactic writings is often very different from the meaning attributed to those same writings by logic.

For instance, let us imagine the following dialogue:

A: Hi, how are you?

B: Not so good, I have a cold.

A: Oh my, at this time *everyone* has a cold!

Before proceeding, the reader is invited to interpret the last sentence from a strictly set-theoretical perspective. What does the speaker A mean by saying ”everyone has a cold”? They most probably do not mean that all human beings have a cold (as a purely logical interpretation would suggest), but they probably do not even mean that *most* human beings have a cold. They simply mean that a *greater number of people than usual* have a cold, a meaning totally different from the logical one. Also consider how these quantifiers—with their ambiguous interpretation—intervene in the construction of a sentence. For example, in Italian, a double negative such as ”non so niente” (literally ”I don’t know nothing”) is absolutely correct and does not lead to misunderstandings. In French, the phrase ”Aujourd ’hui, tous le bus ne circulent pas” (”Today, all buses do not run”) has caused confusion in real-life circumstances, given the possible interpretation of ”Some buses run” (corresponding to shifting the ”not” before ”all buses”) rather than the intended ”Today, no bus is running” [21]. Note that in Italian, negations can be singular or double depending on the order in which a concept is expressed: for example, ”nessuno ha parlato” (”no one has spoken”) is equivalent to ”non ha parlato nessuno” (literally, ”not spoken has no one”), and ”mai ci avrei pensato” (”never would I have thought”) is equivalent to ”non ci avrei mai pensato” (literally ”I would not have never thought”). It is important to note the interesting case of Latin, where the formulation is not dissimilar to that of logical formalism. ”Nemo non haec dixit” (literally, ”no one does not this say”) actually means ”everyone has said this”—whereas ”non nemo heac dixit” (literally, ”not no one has said this”) means ”someone has said this”. Similarly ”numquam non mendacia dixit” means ”they have always lied” [22].

In the context of negation, Khemlani et al. [23] observe the difficulty of forming a mental model of it. As a consequence, people tend to assign a ”small scope” to its meaning, that is, they tend not to contemplate all the possible cases of negation of a given statement. Moreover, the authors state that the very *symbol* of negation can help to form a mental model of it. This is in line with our observations during the experiment with the negation of quantifiers.

The potential conflict between the mechanisms of interpretation of the symbolic mathematical notations and those of natural language is already present at primary school level [24]; consider not only, as shown earlier, the "everyone" used with various ambiguous meanings, but also, for example, the implication "if-then", which in natural language is often interpreted as an "if and only if". This does not mean that we must avoid the link to natural language; on the contrary, our point of view is that an explicit work on quantifiers should start in connection with natural language from primary school. The aim is not to "correct" the ambiguities and underlying meanings of natural language, but to make students more aware and prevent the ambiguity of language from becoming a barrier to the understanding of mathematical statements. And indeed, most of the authors mentioned above identify an early introduction and an explanation of the logic underlying quantifiers as a possible solution to the problems encountered [25]. We believe that expertize in formal logic will support the interpretation of "informal" logical statements common in natural language. We share Grice's opinion [26] who claims that natural logic can be aided and guided by the simplified logic of the formal framework, but cannot be supplanted by it. In our approach, the gradual introduction of symbols that express quantifiers and negation aims to distinguish common language from logical language, analyzing their similarities, differences, and ambiguities.

2.3 Game Semantics

On a formal logic level, *Zermelo Game* fits into the framework of Game Semantics, which is an approach that seeks to give meaning to logical formulas through a game between two players, the Proponent and the Opponent. In this context, the Proponent asserts the validity of a formula, while the Opponent does not agree, trying to prove that the formula is false. The interaction between the two players is guided by the rules of the game, which depend on the structure of the formula and the connectives and quantifiers involved. In Game Semantics, connectives (such as *and* [denoted by the symbol $\wedge$] and *or* [$\vee$]) and quantifiers (*for every* [$\forall$] and *at least one* [$\exists$]) are indeed interpreted in dialogical terms. Intuitively, if the Proponent asserts "$A \wedge B$", the Opponent can choose which between A and B to attack, testing the validity of the joint assertion. However, if the Proponent argues "$A \vee B$", the Opponent asks which of the two options is true and the Proponent is obliged to provide an answer. Regarding quantifiers, Game Semantics adopts an approach similar to that just described. If the Proponent makes an existential assertion, such as "$\exists\, x$ such that $P(x)$", it is their task to provide a "witness", that is, an element x that satisfies the property. For example:

Proponent: There is at least one number that is both even and prime.
Opponent: Which one?
Proponent: 2!

On the other hand, if the statement is universal, such as "$\forall x P(x)$", the Opponent must look for a counterexample, that is, a witness x for which $P(x)$ is not true. For example:

Proponent: All polygons with four sides are squares!
Opponent: That's not true! Look at this rectangle.

Zermelo Game is inspired by Game Semantics, acknowledging the important role of the "witness". As already mention, Game semantics is an interesting field of study because it links the proof of a formula to the notion of winning strategy in a game. In other words, proving a formula means finding a strategy that allows the Proponent to win the game, regardless of the Opponent's moves [27]. In general, in our opinion, viewing proofs as winning strategies in a dialogue captures their true nature: to prove means to convince beyond any reasonable doubt (the "axioms" shared with the interlocutor) that a claim is correct. The dialogic nature of a proof is found in overcoming possible objections and finding values and constructions in a way that satisfies or refutes the statement [28, 29]. This approach provides an intuitive and dynamic perspective on logic and the semantics of proofs, highlighting the interactions between the different elements of the formulas and the strategies used to prove their truth or falsity [30, 31].

3. Design and Implementation

Zermelo Game is a free online educational video game designed to promote understanding and conscious use of the quantifiers *all*, *not all*, *at least one*, and *none*. The goal in the game is to earn as many points as possible within a set time by correctly answering the questions posed.

The game is part of an educational path focused on quantifiers, which aims to develop four fundamental skills: evaluating sentences containing quantifiers, constructing sentences with quantifiers, building sets that respect certain conditions containing quantifiers, and determining the appropriate quantifier to apply to a property given a set of elements. *Zermelo Game* specifically focuses on the last skill, although the others are closely interconnected.

When playing the game, the main goal is indeed to determine whether all or not all elements of a given set enjoy a certain property, or alternatively, whether at least one or none of them do. The game presents various environments that may also require other mathematical skills.

On the home page (Figure **1**), the player (or the teacher) selects which quantifiers to play with (either one or both); the environment, which determines what types of objects will appear on the screen (colors, polygons, numbers, or bags); the level and the time available in the match (we underline that the level is chosen manually by the player or by the teacher). They can also choose the negation or witness modes, which we will describe in Sections 3.5 and 3.6.

Figure 1. The *Zermelo Game*'s homepage: `en.oiler.education/zermelo`

In *Zermelo Game*, the use of symbols to indicate quantifiers is gradually emphasized. In earlier levels, symbols are accompanied by the equivalent expression in natural language, and the teacher and player can ignore them. In later levels, understanding the symbols becomes important, although the teacher will always invite students to consider the linguistic interpretation of the expressions they read. In the BAGS environment, the notation $Y(x)$ is also used to express that the object x has the property Y (*e.g.*, BLUE (ball)).

The levels and gaming environments are intended to progressively develop competencies. The POLYGON environment requires competencies in the thematic core of geometry, the NUMBERS environment in the thematic core of arithmetic, while the COLORS and BAGS environments refer purely to the thematic core of logic. If the player gives the wrong answer, a 'game over' message is displayed otherwise they continue until the time runs out. In both cases they can enter the leaderboard with the score achieved or start over from the beginning.

3.1 Colors

The COLORS environment requires logical skills only, with no other mathematical skills involved. In the only available level, once the two quantifiers (*all* and *at least one*) have been chosen, players need to identify whether all the balls shown are of a certain color (red, green, blue) or if at least one or none of them are of that color (Figure **2**).

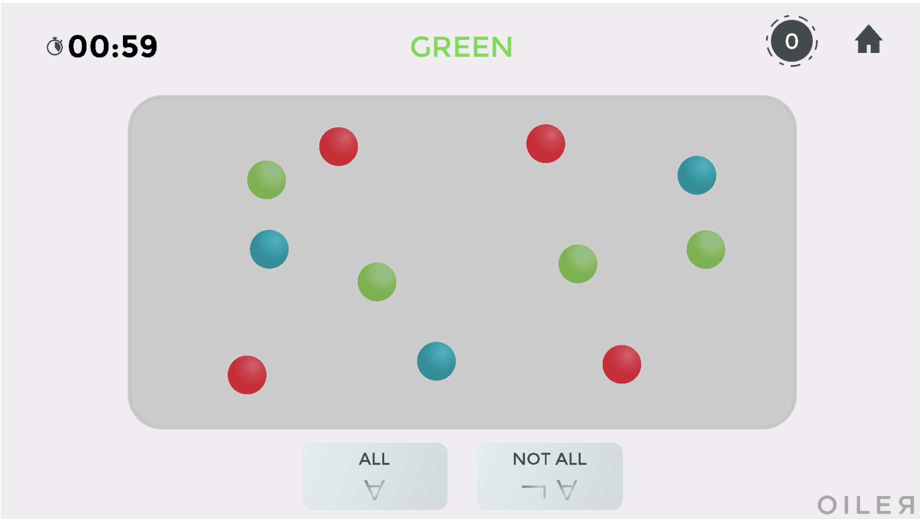

Figure 2. The player is asked whether all or not all of the represented objects are green. The correct answer is that *NOT ALL* balls are green.

3.2 Polygons

In this environment, players need to recognize specific properties of polygons. The levels of difficulty are organized as follows:

In Level 1, the elements that appear are polygons (*i.e.*, plane figures bounded by segments), and the properties are TRIANGLE, QUADRILATERAL, PENTAGON, and HEXAGON (polygons with more than six sides do not appear). For variety, some polygons have shapes that are not typically used in school practice: to respond, it will always be sufficient to count the number of sides (or equivalently the angles) of the polygon in question (Figure **3**). This level can be proposed from the first grade.

In Level 2, in addition to the properties of Level 1, the EQUILATERAL property is introduced. A polygon is said to be equilateral if all of its sides are equal in length. Similarly, a polygon is not equilateral if it has at least one pair of different sides. We note that an equilateral quadrilateral is commonly called a rhombus (from the Greek rhómbos, meaning spinning top).

In Level 3, in addition to the properties of Levels 1 and 2, the properties AT LEAST TWO EQUAL ANGLES, AT LEAST TWO EQUAL SIDES, ONE RIGHT ANGLE, and ONE OBTUSE ANGLE are introduced. The properties "one right angle" and "one obtuse angle" are to be understood as "the polygon has at least one right/obtuse angle". In fact, very often, the expression "at least one" remains implicit. The properties "at least two equal angles" and "at least two equal

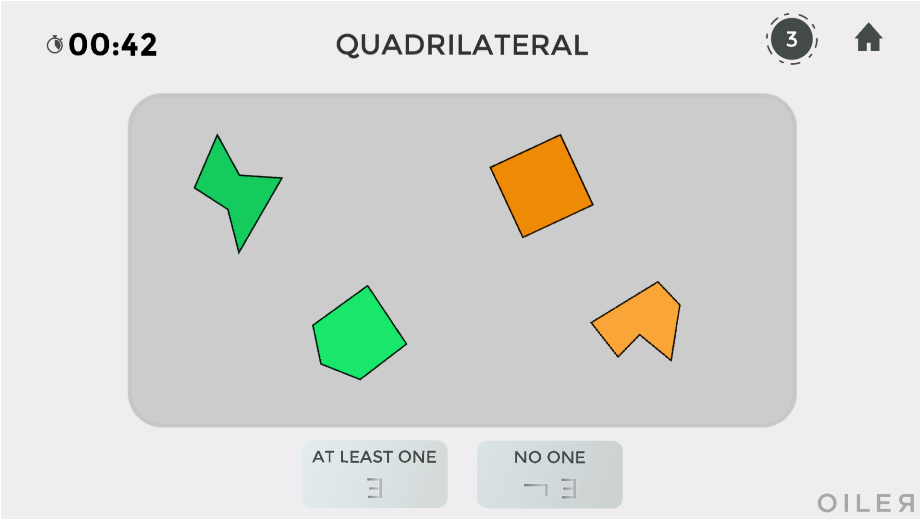

Figure 3. The player is asked whether at least one or none of the represented objects are quadrilaterals. Since there is an orange square, the answer is *AT LEAST ONE*.

sides" appear exclusively in reference to triangles. The class will note that the properties are equivalent: a triangle has at least two equal sides if and only if it has at least two equal angles. A triangle of this type is commonly called isosceles (from the Greek *isoskelés*, where *ísos* means equal and *skélos* means side).

Starting from Level 4, the response buttons change slightly, giving more space to symbolism: in particular, the expression "all" is replaced exclusively by the symbol ∀ and the expression "at least one" is replaced exclusively by the symbol ∃. In this way, we encourage students to move away from phrases toward symbolism.

Here, the teacher can remind the students that the symbol ∀ derives from the English word ***All*** while the symbol ∃ from the English word ***Exists***.

In Level 4, in addition to the previous properties, the property AT LEAST TWO PARALLEL SIDES is introduced. For a triangle, it is impossible to have two parallel sides, while a quadrilateral with at least two parallel sides is usually called a trapezoid (from the Greek *trapézion* meaning *small table*). Finally, Level 5 adds the properties EQUIANGULAR and REGULAR. A polygon is called equiangular when all of its angles are equal, and is called regular when it is both equiangular and equilateral. A quadrilateral with equal angles is commonly called a rectangle (because if a quadrilateral has all equal angles, it consequently has all right angles), while a regular quadrilateral (which is both a rectangle and a rhombus) is commonly called a square.

3.3 Numbers

In this environment, players need to address specific properties of natural numbers. The levels of difficulty are organized as follows:

In Level 1 of NUMBERS, numbers between 0 and 9 appear and the properties EVEN and ODD are used. A given natural number is even if there is another natural number that, if added to itself, results in the given number. For example, 10 is even because $10 = 5 + 5$, while 0 is even because $0 = 0 + 0$. A number that is not even is odd (Figure **4**). In Italian, as in several other languages, the word for odd (*dispari*) comes from a negation of even (*pari*), using the negation prefix "dis-".

More precisely, a number n is even if $\exists x(n = x + x)$. As explained in more detail in Section 2.3, when arguing with a ∃, a witness must always be provided. In particular, it is not enough to

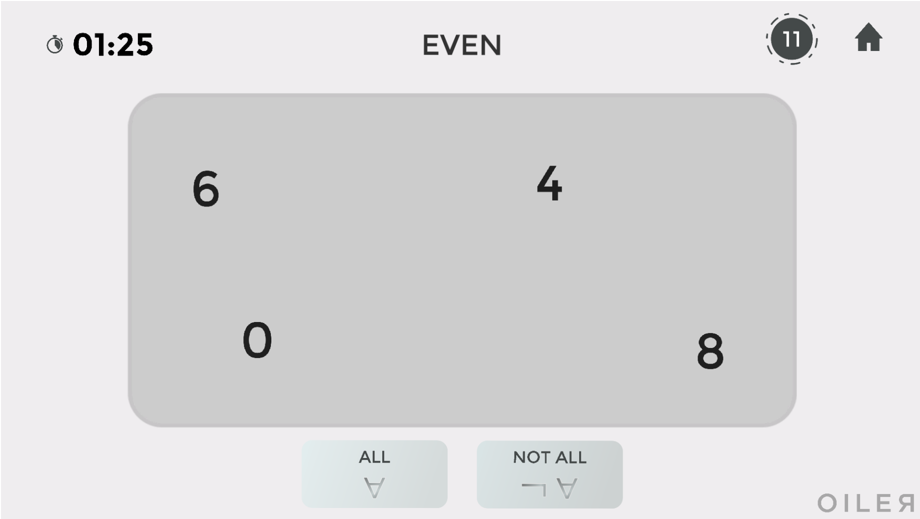

Figure 4. The player is asked whether all or not all of the represented numbers are even. The answer is *ALL*.

say ”6 is even”, but it is always necessary to specify why[3]: 6 is even because $3 + 3$ equals 6.

In Level 2, alongside the properties in Level 1, we are introduced to GREATER THAN and LESS THAN. In this case, the properties extend to numbers between 0 and 100, as shown in Figure **5**.

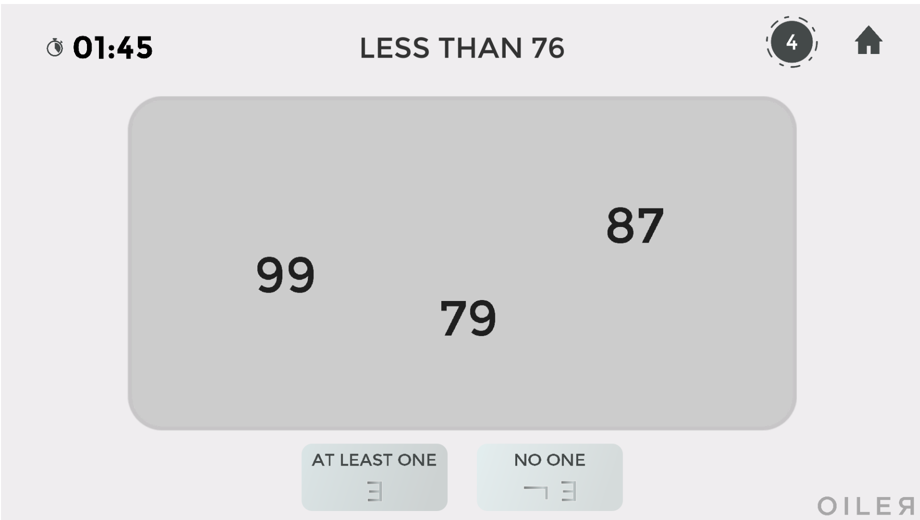

Figure 5. The player is asked whether at least one or none of the represented numbers are less than 76. The answer is *NO ONE*.

In Level 3, in addition to the previous properties, the properties DIVISIBLE BY 3, DIVISIBLE BY 4, DIVISIBLE BY 5, and LAST DIGIT also appear. We note that divisibility is an existential statement: indeed, x divides y means $\exists k(k \times x = y)$. It is therefore appropriate to request a witness from the player stating divisibility.

[3]We should clarify that this dynamic of argumentation on even numbers is not present in the game; we have emphasized it here only to show how the concept of ”witness” is typical in many areas of mathematics.

3.4 Bags

In this environment, alternating quantifiers appear for the first time: the player must consider phrases such as ”all bags contain at least one blue ball” or ”at least one bag contains only blue balls”—that is, phrases with two quantifiers.

A property referring to individual bags appears at the top of the screen. At the bottom, players need to specify whether there is a bag with this property, that is, a bag that has only blue balls (Figure **6**). The answer is negative and therefore they must click on the button ”¬∃ BAG”.

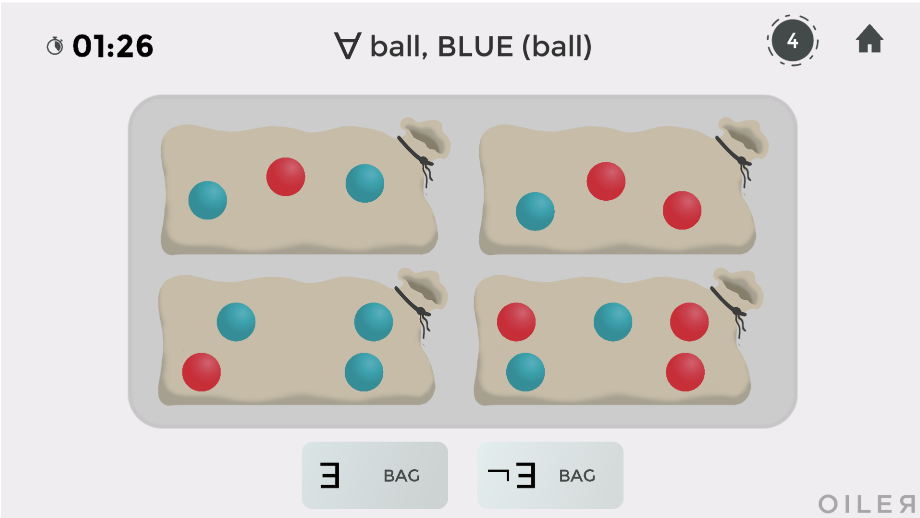

Figure 6. The player is asked whether there exists a bag containing only blue balls. The answer is ¬∃ *BAG*, because no bag contains only blue balls.

As a second example, consider the situation in Figure **7**. The statement at the top describes a property that refers to individual bags: ∃ ball, BLUE(ball), to be read ”there exists a blue ball” or ”at least one ball is blue”. At the bottom, the question is whether all or not all bags verify this property, that is, whether all bags contain at least one blue ball. The answer is negative and therefore the player must click on the button ”¬∀ BAG”.

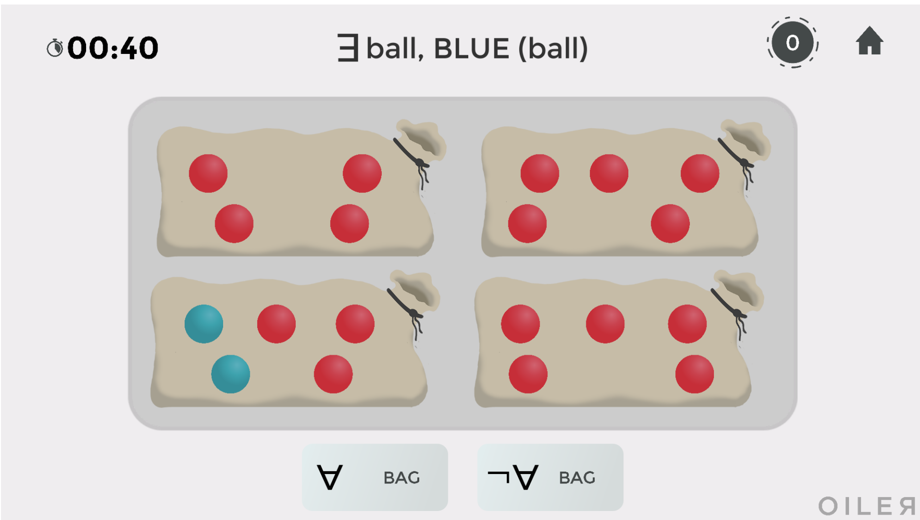

Figure 7. The player is asked whether all or not all of the represented bags contain at least one blue ball. The answer is ¬∀ *BAG*, because three bags have no blue ball in them.

3.5 Witness Mode

If the player selects *Witness mode* on the home page, a new rule is added. If the player clicks on ”not all” or ”at least one” (that is, when they give an existential answer), they must provide a witness, *i.e.*, indicate an object that testifies to the choice made. In the case of ”not all”, players are required to click on an object that does not have the indicated property. For example, referring to Figure **8**, after clicking on ”not all”, the player needs to select a polygon that is not a pentagon—in this case the green rectangle.

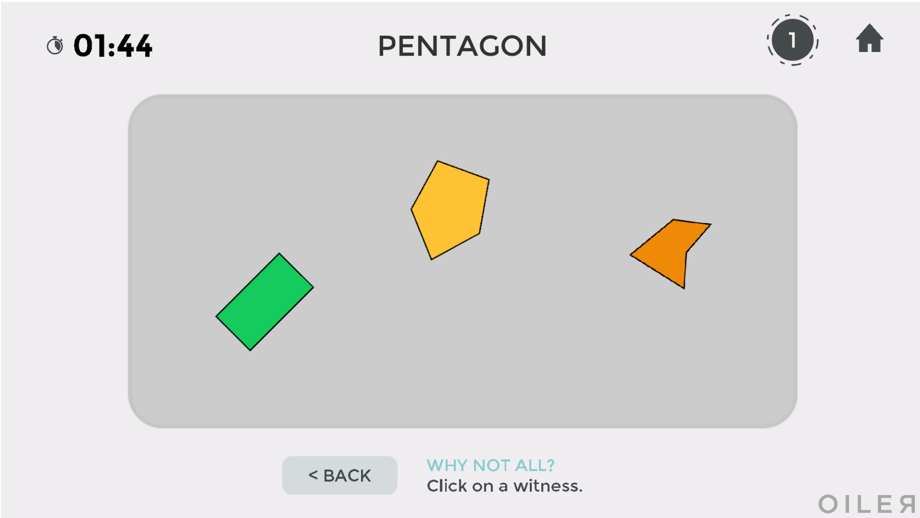

Figure 8. The player is asked for a witness that not all of the represented figures are pentagons. The witness for the answer is the green rectangle.

Similarly, in the case of ”at least one”, players are required to click on an object that has the indicated property. For example, referring to Figure **9**, after clicking on ”at least one”, they must select an even number—in this case, 0.

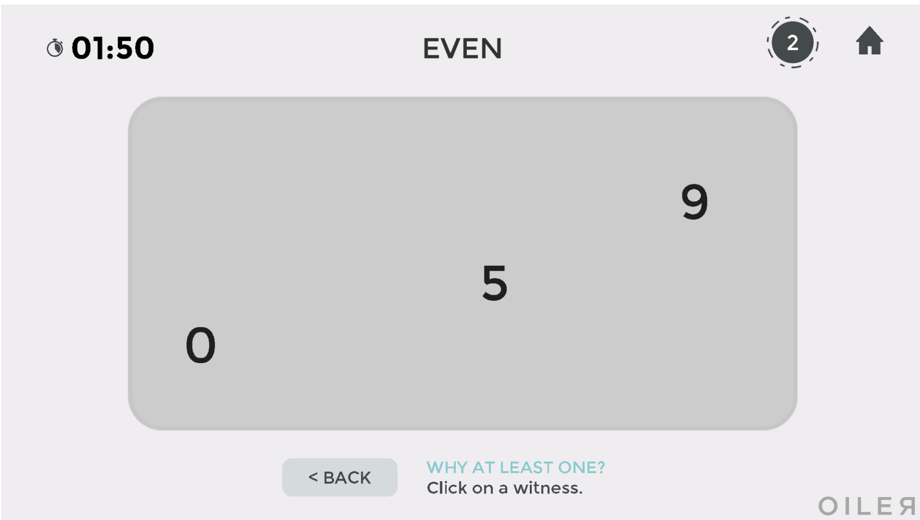

Figure 9. The player is asked for a witness that at least one of the represented numbers is even. The witness for the answer is 0.

The witness mode in the BAGS environment—while following the same principle presented above—is more complex: in this case, a dialogue is initiated between the player and the computer.

The following example is used to clarify this issue.

If the player asserts that there is at least one blue ball in all bags, then they should be able to identify a blue ball in any bag the computer may choose (Figure **10**).

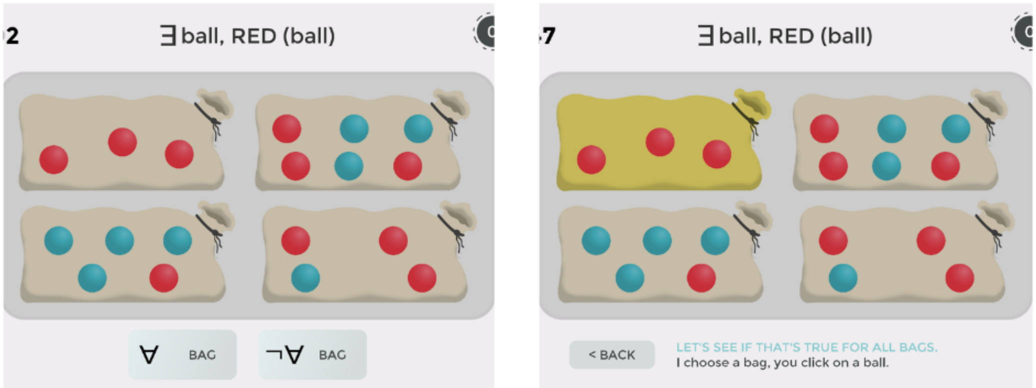

Figure 10. On the left, the player is asked whether all or not all bags contain at least one red ball. The player states that all bags have at least one red ball. On the right, the environment chooses a bag and the player has to identify a red ball in it, *i.e.*, a witness.

Here are some examples of possible scenarios (the fourth corresponds to the situation illustrated in Figure **10**):

- if the player claims that all bags have all red balls, they do not have to do anything (0 total clicks);
- if they claim that not all bags have all red balls, they need to click on a bag where at least one ball is blue, then click on a blue ball (2 total clicks);
- if they claim that at least one bag has all red balls, they need to click on the bag with all red balls (1 total click, made on the bag);
- if they claim that all bags have at least one red ball, they need to click on a red ball in a bag chosen by the computer (1 total click, made on a ball).

3.6 Negation Mode and Its Integration with the Witness Mode

If the player selects the negation mode, then the negation of any properties can also appear at the top of the screen. The symbol ¬ is indeed read as ”not”. For example, being ¬ GREEN means ”not being green”—in the context of the game, this means being either red or blue. Similarly, being ¬ EVEN means ”not being even”, *i.e.*, it means being odd.

In the situation depicted on the left of Figure **11**, a player has to choose between the response that at least one ball is not green, therefore either red or blue, and the response that no balls are not green. If they are also playing in witness mode, they will then proceed to click on a red or blue ball.

It is worth highlighting the tricky case shown on the right of Figure **11**. Being a ”non-hexagon” means, in the context of the game, being a triangle, quadrilateral or pentagon. Therefore, in the right picture of the Figure **11** not all shapes are non-hexagons, because the blue polygon at the bottom right has 6 sides. If playing in witness mode, the player will have to select precisely on that hexagon.

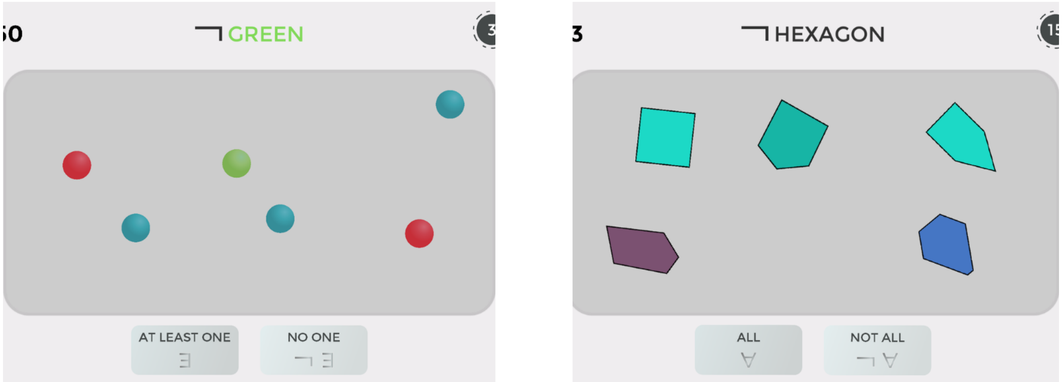

Figure 11. On the left, the player is asked whether at least one or none of the represented balls are not green. The answer is that at least one ball is *not green*, because there is at least one red or blue ball. On the right, the player has to decide whether all or not all shapes are non-hexagons. The answer is *NOT ALL*, because at least one polygon is a hexagon (*i.e.*, the blue polygon in the bottom right corner).

3.7 The Leaderboard

In the game, there are two types of leaderboards: a local leaderboard, which exclusively lists the scores obtained on the device being used, and a global leaderboard, which lists the scores obtained by all users who have played the game. The global leaderboard is readily accessible online, located directly beneath the game interface (see Figure **12**), and updates instantly after each game. To gain access to the global leaderboard, both quantifiers must be selected, as well as the ”witness” mode and the ”negation” mode; moreover, player must set the highest possible level and the gameplay duration to one minute. The global leaderboard is intended to serve as a feedback tool, especially suited to support the motivation and engagement of players. On the other hand, the local leaderboard is accessible directly within the game at the end of each play session and is a useful tool for the teacher to evaluate the class’s progress when students work independently.

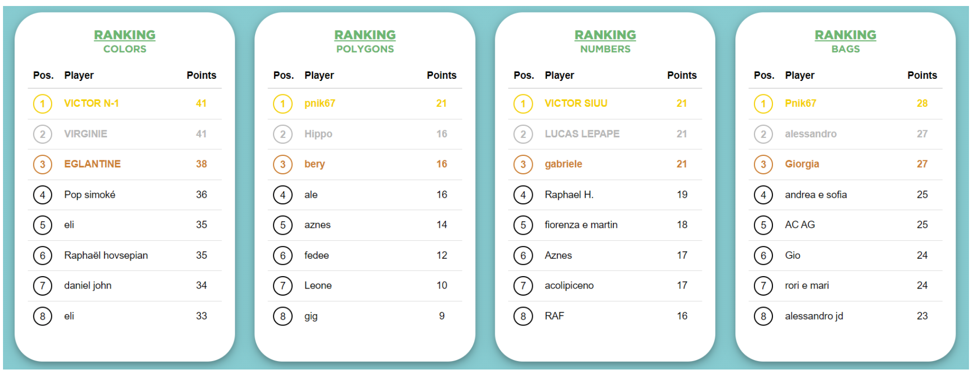

Figure 12. There are four distinct global leaderboards corresponding to four different environments.

4. Evaluation

In this section, we will report results from trial runs with students who used *Zermelo Game*. To frame our findings around the questions raised in the Introduction, we will primarily focus on the COLORS and BAGS environments, as these environments incorporate logical elements without also requiring auxiliary mathematics skills, as is the case for the POLYGONS and NUMBERS environments.

The trials were conducted both in primary school (two classes of Italian schools and two classes of French schools, with pupils aged between 7 and 10 years) and high school classes. Primary school trials were carried out directly in the classroom, in the presence of the class teacher and a researcher. For high school trials, groups of approximately 20 students attended workshops held at the university by the researcher involved in the study. Four workshops were held, with three classes from Italian schools and one class from a French school, each lasting three days. The students ranged in age from 16 to 18 years.

In the French primary schools the game was played in a collective way by projecting it on the interactive whiteboard. In the Italian primary schools the work was carried out as a group in the classroom, as well as in the computer lab, in which students worked together in pairs. In third-year classes the students also competed against one another for higher positions in the game rankings.

High school students worked independently, or in pairs, at the computer, and then explained what they had done to the researcher. They mostly worked within the BAGS environment and competed against one another on the game rankings.

Overall, according to the teachers' and researchers' field notes, all students seemed to enjoy using the software and became comfortable with the game in about half an hour. It is also important to note that no students—whether in primary school or high school—stopped playing out of frustration (owing to not understanding the dynamics or not achieving their set goals).

4.1 Primary School

Before starting to play *Zermelo Game*, the educational path Zermelo included some activities: wearing the masks of a knight or a knave, students were invited to make true or false statements (for instance, ”horses fly”), or to describe objects correctly or incorrectly (”this chair is brown”). This served as an introduction to negation. Subsequently, some tables[4] were shown on the interactive whiteboard, and students again were asked to describe them (for instance, looking at the table in Figure **13** with colored triangles, the students could make the true statement ”there is a green triangle” or the false statement ”there is a square”); sometimes the teacher described the table and the students had to guess if a knave or a knight was speaking. After this first session students were invited to describe the tables using the words ”all”, ”not all”, ”at least one”, or ”no one” (for instance, ”all the shapes are triangles”).

In the second session, *Zermelo Game* was introduced. The introduction occurred gradually, with the difficulty increasing in each session. The first game was played with the $\forall$ quantifier only and on Level 1 of the COLORS environment, with neither negation or witness mode. To illustrate the game, the teacher initially explained how to interpret and read the various elements that appeared on the screen (the property at the top, the set of objects, the answer buttons below) by providing three or four examples. However, the teacher did not provide the answers to the class in any case. Instead, the students were asked to respond collectively, not by calling on any particular student, but by answering in unison. During this phase, the teacher was not only reading

[4]Zermelo's tables represent sets of objects (such as people, animals, figures, etc.) and can be found here `https://en.oiler.education/school/materials/tables/zermelo`

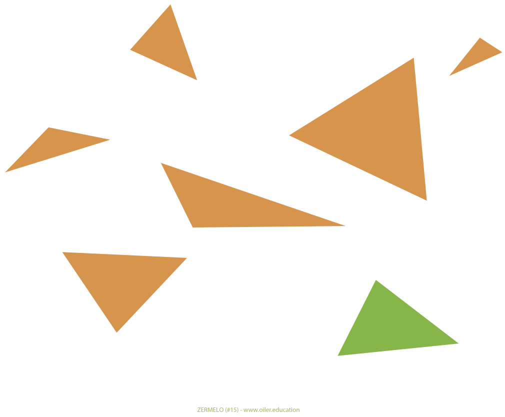

Figure 13. A Zermelo table with some triangles.

and articulating the question but also rephrasing the class's response with correct verbalization, paying close attention to the language used. Subsequently, the students were called one by one to answer a specific question: this time, the articulation of both the question and the answer was handled by the student responding at that moment, while the teacher intervened to correct any errors or to repeat a phrase already correctly formulated by the student. The students were then asked to provide reasons for their answers.

In these first two phases, although the score and the timer were present and perceived by the class as elements of the game, they were not deemed important. In fact, the teacher always selected the maximum time available and did not give any weight to the total score accumulated up to that point. Once the context of the game was clear, all students were able to answer the questions correctly. Although Level 1 of the COLORS environment eventually became easy for the class, this does not diminish its didactic value: firstly, a thorough understanding of the mechanisms in play is essential to move onto the more difficult levels with confidence and awareness. Secondly, working at a level at which students feel comfortable improves not only the accuracy of their answers but also their speed. This acceleration undoubtedly shows students' understanding of the mechanisms used to generate these answers. As already mentioned, students discussions were recorded, to try to extract out key points of their use of quantifiers and negation. The most interesting points refer to their reasoning: not the answer the student gave, but the reason behind it. For example, when asked to discuss a turn such as the one illustrated in Figure **14** (left side), we recorded the following three conversations.

student A: Not all of them are blue.
teacher: Why?
student A: Because *some are blue*!

student B: Not all of them are blue.
teacher: Why?
student B: Because *only some are blue*!

student C: Not all of them are blue.
teacher: Why?
student C: Because *some are green or red*!

In these three dialogues, the teacher assumes the role of the Opponent, which will later be

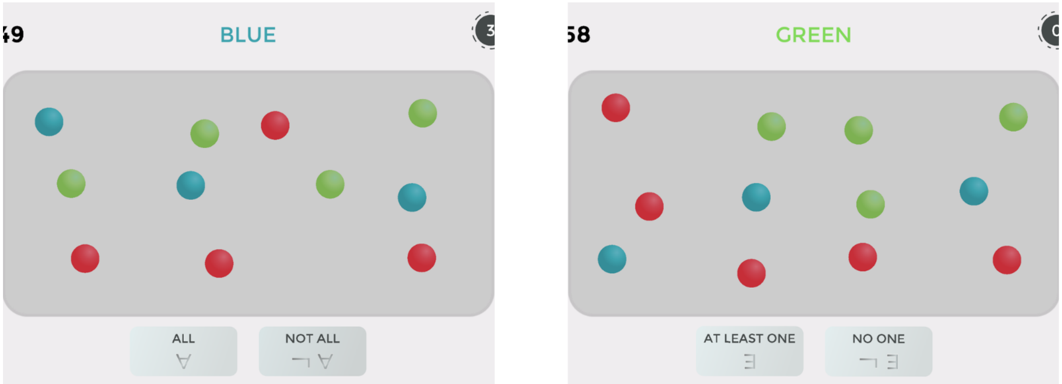

Figure 14. On the left, the player is asked whether all or not all balls are blue. Even though answering the question is not difficult, providing the right argumentation is essential to move onto the more difficult levels. On the right, the player has to decide whether at least one or none of the balls is green. The answer is that at least one ball is *green*.

taken over by the game. In the first two exchanges the focus is on the blue balls (even though the second answer provides more precise information), whereas in the third exchange the focus is on the balls that are not blue, ignoring the blue balls entirely. Initially, almost all students behave like students A or B, with only a very small portion able to argue correctly like student C right from the start. This situation is of interest because it highlights the difficulty present from early years of education in proving the falsity of a universal quantifier—in other words, when trying to come up with a witness that does not have the property (a counterexample) as opposed to one that does (an example). As a similar difficulty is not encountered when discussing normal existential quantifiers, it seems that the cause of this struggle lies firmly in the negation, and perhaps in the scarce attention paid to this concept at school, as outlined in the literature presented in Section 2.2 (see *e.g.*, [10, 25]). Indeed, if we limit ourselves to working with true statements or, more generally, with objects that satisfy a given property, we lose the opportunity for full understanding: to understand the concept of *red*, it is important to question what *not red* means; to understand the concept of *triangle*, it is important to question what *not triangle* means; to understand what *greater than* 3 means, it is important to question what *not greater than* 3 means (which is not less than 3!).

Referring again to Figure **14** (left side), the fact that some balls are blue does not help us to reach the correct answer, as the answer would not have changed had there been many more blue balls or indeed no blue balls at all: *not all balls are blue because at least one is either red or green*.

The next step in reasoning with the universal quantifier consists in shifting focus from the set of red and green balls to one particular red (or green) ball: not all balls are blue because *this one is red*. As shown earlier, the ”witness” setting in *Zermelo Game* asks players to do precisely this by clicking on a ball that does not satisfy the property. Independent play using this setting helps students to understand and accept the role of the witness. This understanding in turn improves their subsequent reasoning and helps to develop the meaning of the universal quantifier. Indeed, during sessions of independent play, it is common to catch students talking to themselves: verbalizing the situation proposed by the game seems to aid their performance (see also Section 5).

In the following session, the game was played exclusively with the existential quantifier ∃.

When looking at the example illustrated in Figure **14** (right side), we find once again that successful reasoning involves moving from ”there are some green balls” to ”this ball is green”. This is likely due to the fact that the universal quantifier had been discussed previously, and that these two quantifiers are fundamentally connected.

Consequently, both quantifiers began to be used simultaneously, integrating the witness mode. This mode represents the incorporation of argumentation dynamics within the game, an aspect to which the teacher had devoted considerable attention. In schools equipped with a computer lab, this was the first activity for pairs of students working on a single computer, clearly emphasizing the competitive goal of scoring. Although no recordings of student dialogues exist, the dynamics between pairs were keenly observed by the teacher and researchers. Notably, the class had effectively adopted and internalized the argumentation through the witness mode, with children pointing out specific balls to persuade each other, as illustrated in Figure **16**. Let us emphasize that the local leaderboard made it much easier for the teacher to identify which pairs were having the most difficulty, intervening where appropriate.

After the class had gained a certain understanding of the game dynamics, also through exploring Level 1 of both the polygon mode and the number mode, the symbol ¬ was introduced at the beginning of the next session to represent ”not”. Students were explained that, within the context of the game, being ¬RED means to be either green or blue. Similarly, being ¬BLUE means to be either green or red, and being ¬GREEN means to be either red or blue.

The greatest difficulty for almost all students was identifying an appropriate witness in a situation like the one shown in Figures **15**.

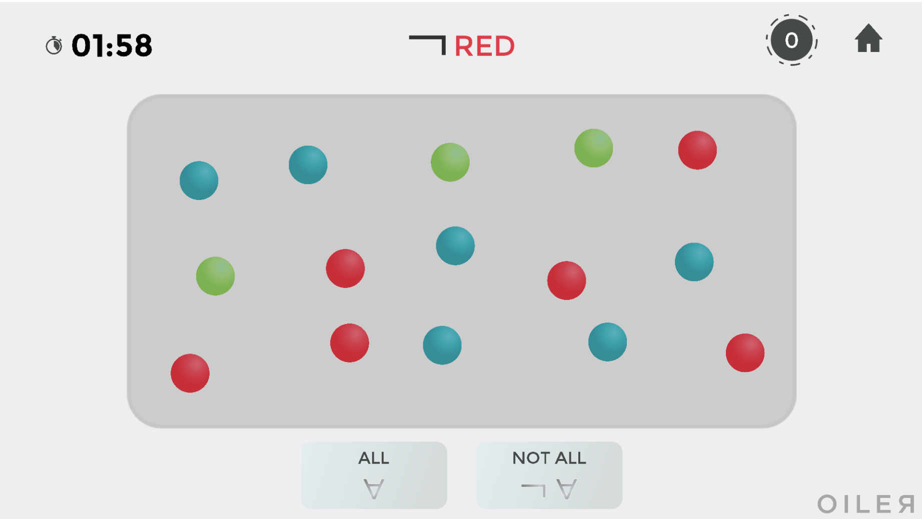

Figure 15. The player has to decide whether all or not all balls are not red. The answer is that not all balls are *not red*, because there is at least one red ball.

An appropriate witness here is a red ball, because if not all balls are not red, then at least one is red. This way of using a quantifier between two negations is not only common in mathematics, but also in natural language. The high school students faced similar difficulties, which we discuss in the following section.

4.2 High School

During the three-day workshop, students engaged in various activities about connectives, quantifiers, logical language[5], and other activities involving games and winning strategies. Specifically,

[5]Not just in its relationship with mathematics but also with everyday language.

Figure 16. Primary school students playing *Zermelo Game* in the ”*not red*” condition. One of the students is about to point out a specific ball to convince the other student.

four workshops were conducted, involving a total of 88 high school students, of which 25 were French and the rest Italian. The classes were often mixed, with students of different ages and from different schools. Throughout the workshop, *Zermelo Game* accounted for about 35% of the activities and was introduced almost at the beginning.

In this case, the researcher was also responsible for filming, so only specific scenes were recorded. The presentation dynamics of the game followed the same patterns used in primary school, albeit at a different pace. The students quickly started playing in pairs, with the teacher instructing them on which modes and levels to select. In this case as well, extensive discussion about the game was observed among the pairs. Although the high school students played with all environments in *Zermelo Game*, we describe here their experience with the BAGS environment, as it best reflects the way students faced alternating quantifiers.

Here we examine a discussion between a researcher and a student, in which the student explains their reasoning to the researcher (see Figure **17**). Translating the formal notation, the question states ”all the balls are blue”, with a choice between ”a bag exists” and ”a bag does not exist”. Upon choosing the answer $\neg\exists$ BAG, the student defends their choice by saying ”No, because here there is a red ball”. Although the answer is correct, the reasoning is flawed, or at least incomplete.

With the ”in this bag” being implicitly understood, that ”here” uttered by the student implies that the bag used to defend the answer can be chosen by the player. Instead, the bag is chosen by the Opponent (Figure **18**). Therefore, when the Opponent puts student’s affirmation to the test by choosing a bag, the student need to explicitly indicate a red ball, thus showing that not all balls within that bag are blue. Indeed, stating ”there is no bag in which all balls are blue” is equivalent to stating ”in every bag there is at least one red ball”.

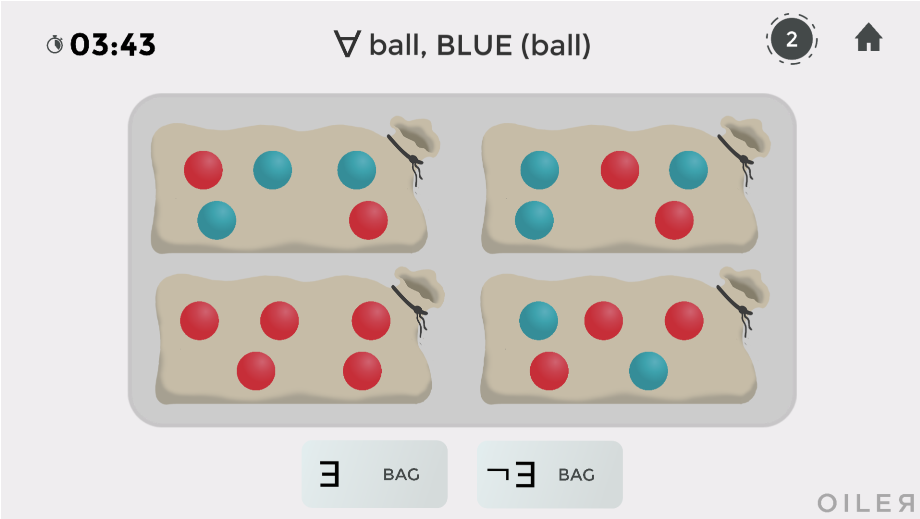

Figure 17. The student has to decide in which bag there are only blue balls. The answer is ¬∃ BAG because no bag has only blue balls in it.

It is worth highlighting here that the last step of this reasoning, where students show that not all balls are blue by choosing a non-blue witness, is similar to that used in the COLORS environment. The BAGS environment is an environment that contains colors within it, but the statements made refer to sets of colors. The student's reasoning is therefore incomplete, or in some way implicit. A more comprehensive explanation would have been something along the lines of "If the computer chooses this bag, then the witness will be this ball [clicking on any red ball within that bag]".

Moving onto another exercise, the same problem comes up. "Is there a red ball in every bag?", the game asks. "Yes, there it is!", replies the student. In this case as well, the role of the Opponent is implicit. At no point has the student mentioned the fact that the computer has made a move in choosing a bag.

When, instead, the two choices are made by the student (*e.g.*, in the case of two existential quantifiers), then both the choice of bag and ball is explicitly mentioned by the student: "No, because in this bag, for example [first click], there is a blue ball [second click]". This is probably because the act of clicking on each element encourages the student to reason at each step. This aspect of Game Semantics is essential to understanding the concept of proof, and here we highlight the difficulty students may face in reasoning correctly. The game automates this process, and introduces it to the students. Indeed, after gameplay, the role of the computer in the BAGS environment was discussed and analyzed.

Despite being fluent in mathematics, the student once again proposes a dynamic that appears to be shared by many others: to have little familiarity with argumentation on quantifiers, and thus with quantifiers in general. Our hypothesis is that the use of *Zermelo Game* will help build this familiarity owing to the various dynamics discussed previously.

StudentD: There is a red ball in every bag, so I will choose a red ball.
Teacher: In a bag...
StudentD: ... that the computer has chosen.

In this case, the student has been pointed in the right direction and has concluded their reasoning correctly.

These reported dialogues are representative of most students. Indeed, the difficulty in identifying the appropriate witness, as well as understanding the natural role played by the computer as the Opponent, was typical. However, after a few games, students found it easy to study the

Figure 18. If the student answers the question illustrated in Figure **17** correctly ($\neg\exists$ *BAG*), a bag is selected by the environment and the player must show a red ball in it, proving that not all balls are blue in that particular bag.

dynamics of the Opponent[6], eventually seeing them as generalizations of the dynamics of the COLORS environment, which the students had previously understood and accepted.

We now aim to analyze the questionnaires that students completed anonymously at the end of the experience. A questionnaire was administered to 22 students asking what they liked most, what they liked least, and what they felt they had learned. For the remaining students, in addition to questions about what they liked most and least, they were explicitly asked for their opinion on symbolism. Overall, the evaluation of the educational path was positive in every questionnaire.

In the anonymous questionnaire, there was no explicit mention of *Zermelo Game* in the questions; therefore, any references to the game in the responses were completely voluntary. In general, when responding to the question ”What did you enjoy the most during the three days together?”, some students generically mentioned that they enjoyed the entire program; many stated that they appreciated the group work and collaboration in general, building and then playing with the Tower of Hanoi, or finding a strategy in the Tic-Tac-Toe game.

The most interesting aspect related to *Zermelo Game* is that 56 out of 88 students explicitly mentioned computer games among the activities they enjoyed the most. We believe these numbers cannot be used for a quantitative analysis because the sample size is too small to draw quantitative conclusions, and because there were no specific questions about the game. The fact that students enjoyed the game does not exclude that they also enjoyed other parts of the programme; it is also plausible that those who simply stated they enjoyed the entire program also enjoyed the game.

Here we report feedback from students who explained why they enjoyed the game. A first category of responses pertains to the game as an aid to reasoning.

> *The Zermelo color games are very fun and help me stay focused.*

> *Among all the activities we did, the most interesting and engaging for me were the games we played with Zermelo Game and Bul Game*[7] *because they tested our reasoning abilities and speed in doing so.*

> *Probably the thing I appreciated the most was explaining the topics through the*

[6]Students were asked to describe all possible cases in the BAGS environment, highlighting the type of dialogue that occurred between the player and the computer in each case.

[7]Bul Game is another game related to connectives, but it was used only once, so for much less time compared to Zermelo Game.

> *Zermelo game, making the explanation and acquisition of concepts easier and faster for us students.*
>
> *Learning to use logic while having fun with a very nice instructor. The use of Zermelo Game greatly helped in understanding the material covered.*
>
> *Another thing I appreciated was Zermelo Game because it was useful for better understanding mathematical symbols and thus mathematical writing.*
>
> *I greatly appreciated the approach used during the lessons. In my opinion, using interactive games and fostering healthy competition to explain even complicated topics is a winning strategy.*
>
> *The thing I liked the most was the lab on Tuesday where we did the challenge with the computers with my classmates.*
>
> *The thing I liked the most was the computer games because I had to apply what I learned, and it also created a competition to get on the leaderboard.*

As we can see, the last few students appreciated the competition due to the presence of the leaderboard. For many, the experience was particularly significant for the collaborative work with other students.

> *The things I appreciated the most were the interactivity and the competitiveness. The interactivity because we played many interactive games and always discussed the symbols. The competitiveness, so to speak, because in the interactive games there is a desire to surpass others, which motivated us to improve ourselves and our understanding of the symbols.*
>
> *What I liked most about this experience was the activity involving math games played in pairs, which was very useful and fun.*

Finally, let us look at some responses the students gave regarding the use of symbols. The key aspects to highlight are twofold: on the one hand, the students see a strong connection with natural language; on the other hand, they recognize the utility of symbols in mathematics.

> *Until now, I did not understand their use very much, but now I am aware that they are also commonly used and can be helpful in everyday life.*
>
> *I believe they are really interesting and* beautiful to understand. *I've realized how, thanks to them,* it is always possible to simplify and make things understandable to everyone*, and especially how they are closely linked to everyday life.*
>
> *I discovered that symbols are present in the sentences we utter daily, and therefore, in part, they are not difficult to understand. I find that their use in mathematics is aimed at simplifying complex sentences or conditions.*
>
> *I think they're very useful, both for mathematics and beyond. In fact, during these two days, it was very interesting to learn about the* close link between mathematics and real life through these symbols.
>
> *Useful for training the brain and* understanding Italian.

> *I had never thought that in our daily lives we often use symbols without realizing it, and this also* makes me more aware of what we say and how our language works.

> *Not knowing these symbols before taking this course, I had no idea how important and present they are in our daily lives, especially for simplifying and synthesizing sometimes very complicated concepts.*

> *The use of symbols is certainly very useful, they promote* direct communication free of possible contradictions and misunderstandings*, which is essential in mathematics.*

> *It's a type of language not common to us that, however, turns out to be fun to use also for better understanding of* meanings and negations of sentences in Italian.

Another interesting type of response recognized symbols as having a role of universality, independent of the spoken language.

> *I find that the inventor of these symbols was very intelligent because they allow expressing logic without going through French.*

> *I find that the use of these symbols is very important because it's a* universal language that everyone understands.

5. Discussion and Limitations

Let us revisit the research questions presented in the introduction, starting with the role of the game within the educational path.

Firstly, as specified in the introduction, *Zermelo Game* aligns with the principles presented in Kiili, Devlin, and Multisilta [1]: it is based on sound theoretical principles that are directly integrated into the game, fully utilizing the possibilities offered by technology. Secondly, the game is well integrated within the educational path: the teacher guides the students, leading them to the theme of the game, presents the game in an active and participatory manner, and then follows up the gameplay phase by asking appropriate questions to guide the students' learning (see [32]). At this stage of the research, it is therefore not possible to separate the results related to the game from those related to the entire educational path, except through the literature: even though we do not have objective evidence on the effectiveness of logic games presented in Section 2.1, we can refer to specific research on learning mathematics through game playing [33, 34] and to the wide literature review of the educational effectiveness of serious games proposed by Belotti *et al.* [35]. Our hypothesis on the game's effectiveness is further supported by the responses to the questionnaires administered to high school students. Not only do most students report enjoying the game, but they also find it helpful for reasoning, better understanding the concepts learned as well as their own mistakes, and comprehending the meaning of quantifiers and their associated symbols.

As already mentioned, the role of the game in reinforcing the learning of the educational path should be verified using a media comparison research design (see [36]) through pre- and post-tests and control classes. These were not present in our trial run.

It is also not possible to make definitive statements about the comparison between playing *Zermelo Game* collectively on the interactive whiteboard and playing individually or in pairs. However, without the paired phase, it would not have been possible to observe many of the phenomena reported here, such as the use of the witness as a convincing tool within the community.

Additionally, in the collective gameplay, the competitiveness appreciated by some students in their questionnaires is absent.

Another important research question concerns the role of verbalization within the game. Indeed, one of the most significant findings of the trial runs, in our opinion, are the collaborative, reflective processes supposedly promoted by the game, *i.e.*, the discussions among students themselves as well as between teachers and students regarding why a certain statement is true or false. Students exercise logical reasoning, and they do it explicitly, by verbalizing it. Additionally, the observation of students talking to themselves seems linked to this point: verbalization apparently helps them to reflect upon and clarify the problem they are trying to solve. Moreover, the questionnaires indicate how much high school students appreciate interacting with others and explaining their reasoning.

Different experiences on the role of social interaction when playing a video game can be found in the literature. With reference to a single-player 3D architecture game in which students use mathematical skills to construct a certain structure, Moon and Ke [37] negatively correlate students' peer interaction with task efficiency and learning engagement. On the other hand, in a game in which students have to direct a virus to evade cell defenses, Corredor [38] observes positive effects of peer interactions on students' engagement and performance. According to Corredor, students using the video game produced high-quality conversations focused on disciplinary content: conversations focused on subject matter are generated when game content is associated with disciplinary ideas and representations. In our opinion, *Zermelo Game* is more aligned with Corredor's position. Indeed, it has the particularity of being a game about argumentation, and efficiency in the task should not be understood solely as the score obtained, but as the correct argumentation provided to the reference community (the other students or the teacher). Therefore, while it may be true that higher scores on the leaderboard are achieved individually and without excessive verbalization, the same cannot be said for the learning experience. For instance, in a situation of guided discovery, [39] found a positive correlation between verbalization and the teaching of concepts of logic, including quantfiers. An important work linking thinking, communication, and cognition is one by Sfard [40], which not only considers discourse as a medium of learning, but imagines thinking as communication, and learning as participating in discourse. The so-called commognition paradigm by Sfard could help to interpret the use of the language of logic in classroom discussions and also students' thinking aloud. Byrd *et al.* [41] study the effect of thinking aloud on reflection test performance: we are not sure that what we observed can be framed in thinking aloud protocols. Instead, our observations could be linked to particular theories of learning, such as the ICAP framework. Within this framework, Chi and Wylie examine the role of constructive behaviors such as self-explaining and other forms of dialogue [42] in facilitating learning.

As for the educational value of theoretical elements like the witness, we have seen it "in action" not only in teacher-student dialogues but also in pair work when students explain their answers to each other. It is particularly interesting to note that often, despite providing the correct answer, the correct witness was not identified (particularly in more difficult cases involving negation). This aspect is important because, without the witness mode, the comprehension issue would never have emerged. In other words, it seems that the witness mode is capable of promoting a conceptual change, in the sense proposed by Merenluoto [6], which would otherwise have remained unexplored.

We therefore consider the 'witness mode' an important feature of the game and a crucial part of the learning process. We did not find studies concerning this point, with the exception of the mentioned works on Game Semantics [29, 31] presented in Section 2.3, which do not explicitly cite the role of the witness and are not aimed at primary school. Comparing the motivational and cognitive effects of using or not using the witness mode could significantly contribute to the general understanding of the process of argumentation. This could promote value-added learning research [36]. Here too, control classes are necessary, as well as a broader demographic constitution of the sample.

For future developments, the same kind of controlled experiments could be made in relation to the role of elements such as the leaderboard. According to Park and Kim [43], leaderboards assist users with goal setting, boost competition, and provide feedback. They can help to enhance learning and motivation in a gamified educational environment. Indeed, both primary and secondary students seemed to enthusiastically embrace the competitive aspect it offered, as confirmed by the field notes of teachers and researchers and by some of the responses to the questionnaire administered to the secondary school students. Observers noted that, after about half an hour playing, students achieved good scores[8]. Park and Kim [43] could help us in interpreting the role of the leaderboard of *Zermelo Game* in assessing learning.

The final research point broadly addresses the teaching and learning of logical quantifiers and their symbols, discussed in Section 2.2. In this context, it is essential to mention the questionnaires. All surveys were completed anonymously, and we now summarize the most common responses. Generally, the concept of interaction frequently emerge in the surveys, along with an appreciation for group work, which evidently some students are not accustomed to.

Regarding symbolism, the surveys highlighted the universality of symbols, with the term ”universal” appearing five times within the surveys. Symbols are generally perceived as closely linked to language and reasoning, useful both for their conciseness and for their lack of ambiguity. Many students noted that although symbols are not adequately addressed in schools, they are still used. Our hypothesis, supported by the survey results, is that symbols are not necessarily seen as irrelevant abstractions. If they are perceived this way, it is likely due to a lack of proper introduction. We hope, as is already the case in some high schools, that mathematics education will consistently involve a serious discussion on symbols and their connection to language. Furthermore, as has been extensively discussed, we hope that symbols can be introduced before students reach high school.

The main difficulty identified relates to nested quantifiers (as suggested in [17]), negation (as suggested in [23]), and finding the appropriate witness in universal statements about a certain property, especially when negated. As far as we know, this last point is not present in the literature.

Regarding primary school, according to the field notes, most students enjoyed working with symbols (both in the context of the game and in other contexts), which were presented as symbols used by knights and knaves on their island—this is to clarify the context of the symbols’ usage. In interviews conducted after the lessons, teachers approved the use of symbolism as well, emphasizing that when strange symbols are used in other subjects—such as in history when referring to ancient alphabets—the students are always curious to learn them.

6. Conclusion

The *Zermelo Game*, as introduced in Section 3, is the result of a lengthy evolutionary process. The dynamics and gaming environments have undergone substantial changes following pilot playability tests conducted with classes, individual students, or teachers. Initially, the game lacked the negation mode and featured only the COLORS environment. The POLYGONS and NUMBERS environments were later developed upon considering that the game could also convey significant knowledge in other mathematical areas, and especially that including these new environments

[8]For instance, 30 is an excellent score, because it means that a correct answer is given, on average, every two seconds.

would enhance the logical component of the game. Indeed, the COLORS environment risked keeping the gameplay too abstract, not providing the right perspective and depth to the argumentation dynamics presented, which are indeed applicable in any setting. The negation mode was added because there was a notable difficulty among students in understanding the concept of negation, confirming the assertions made in [23]. Although negation was already implicitly present[9], it was deemed that making it explicit could aid in a deeper understanding of the argumentation dynamics. Finally, the BAGS environment is a natural evolution both from a theoretical and pedagogical perspective. Once it was confirmed that the game appeared to function effectively in classroom settings, it was decided to address the typical difficulties associated with alternating quantifiers.

The *Zermelo Game* was not intended to function as an independent educational activity, but the research direction undertaken in this work aims to demonstrate that it could prove beneficial if integrated into an instructional approach for introducing logic. *Zermelo Game* is not straightforward at its advanced levels, and both students and educators require time to devise the correct answer. We do not know—at the current state of our studies—if the game can assist in preserving the learned concepts. However, the game favors an almost spontaneous verbalization, thereby keeping the connection between language and concepts alive and showing how the underlying meanings are indeed understood. Additionally, as demonstrated in our experience, *Zermelo Game* can also be used in primary school and is one of the few games that accurately transposes formal logic to this educational level. Regarding experiences in primary schools, in two schools where the computer lab was not available, the game was played exclusively on the interactive whiteboard, with the students taking turns. Although this initial phase is useful for introducing the game and its various dialogue dynamics, we believe that the subsequent experience of having the class play in pairs, each with their own device, is essential from an educational perspective.

We hope that in the future, by conducting targeted research, *Zermelo Game* can contribute to the discussion regarding the relationship between verbalization and game-based learning, particularly in the context of logic education.

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Conflicts of interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[9]For example, to assert that not all balls are blue means to assert that at least one ball is *not* blue.

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